JOURNAL OF APPROXIMATION THEORY **83**, 342–346 (1995)

On the Approximation of Positive Functions by Power Series

ULRICH SCHMID

Mercedes-Benz AG, OD/vsk, 70567 Möhringen, Germany

Communicated by Doron S. Lubinsky

Received September 7, 1994; accepted November 26, 1994

The problem to be studied goes back to a question of Erdös and Kövari, concerning functions $M(x)$, $x \in R_0^+$, which are positive and logarithmically convex in $\ln x$. The question to find necessary and sufficient conditions for the existence of a power series

$$N(x) = \sum c_n x^n,\ c_n \geqslant 0 \text{ with } d_1 \leqslant M(x)/N(x) \leqslant d_2,\ x \geqslant 0, \text{ where } d_1, d_2 \in R^+,$$

has been treated by several authors. The present paper concerns a generalization of this problem regarding positive functions $h(x)$, $x \in R_0^+$.

1. INTRODUCTION

By Erdös and Kövari [1] the problem was posed to establish criteria which for a positive function $M(x)$, $x \in R_0^+$, logarithmically convex in $\ln x$ (i.e., $F(t) = \ln M(e^t)$ convex in $t \in R$) ensure the existence of a power series $N(x) = \sum c_n x^n$, $c_n \geqslant 0$, with

$$d_1 \leqslant M(x)/N(x) \leqslant d_2, \qquad x \geqslant 0,$$

where $d_1, d_2 \in R^+$ (notation $M \sim N$).

$N(x)$ shall not represent a polynomial; i.e., $M(x)$ satisfies the condition $x^n/M(x) \to 0$ for $x \to \infty$, $\forall n \in \mathbb{N}$ (M rapidly growing).

In [3, 6, 7] necessary and sufficient conditions were investigated. In [6] the author proved the following

THEOREM 1. *Let $M: R_0^+ \to R^+$ be logarithmically convex in* $\ln x$ *and rapidly growing. With $F(t) = \ln M(e^t)$ and $F_r'(t) = \lim_{h \to 0,\, h>0} (F(t+h) - F(t))/h$, $t \in R$, let the sequences a_n, $\bar{a}_n$ be defined by*

$$a_n = \inf\{t \in R,\ F_r'(t) \geqslant n\},$$

0021-9045/95 $12.00

and

$$\bar{a}_n = F(a_n) - F(a_{n+1}) + (n+1)a_{n+1} - na_n, \qquad \text{for} \quad n \in \mathbb{N}.$$

Then a necessary and sufficient condition for the existence of a power series $N(x) = \sum c_n x^n$, $c_n \geq 0$, *with* $M \sim N$ *is that*

$$F(\bar{a}_n) - F(a_n) - n(\bar{a}_n - a_n) \leq c, \qquad \forall n \in \mathbb{N},$$

for a constant $c \in R^+$.

In the present paper, more generally for a given positive function h on R_0^+ rapidly growing, a necessary and sufficient condition for the existence of a power series $N(x) = \sum c_n x^n$, $c_n \geq 0$, with $h \sim N$ is established (Theorem 2). By this theorem it will be possible to extend certain properties of power series with non-negative coefficients to positive functions, for example to weight functions in "weighted approximation by polynomials" [2, 4, 5, 7]. Theorem 2 also yields a result on maximal power series (Theorem 3).

2. Power Series with Non-negative Coefficients

THEOREM 2. *Let* $h: R_0^+ \to R^+$ *be rapidly growing. Set*

$$u_n = \sup_{x \geq 0} x^n/h(x), \qquad n \in \mathbb{N}$$

and

$$P(x) = \max_{n \in \mathbb{N}} x^n/u_n, \qquad x \in R_0^+.$$

A necessary and sufficient condition for the existence of a power series $N(x) = \sum c_n x^n$, $c_n \geq 0$, *with* $h \sim N$ *is the validity of*

$$h(x)/P(x) \leq c, \qquad x \in R_0^+, \tag{1}$$

for some constant $c \in R^+$.

Proof. We define $M(x) = \sup_{k \in K} k(x)$, $x \in R_0^+$, where $K = \{k: R_0^+ \to R^+,$ $k(x)$ logarithmically convex in $\ln x$ and $k \leq h\}$. $M(x)$ is the greatest minorant of $h(x)$ which is logarithmically convex in $\ln x$. Further we define for $t \in R$, $F_1'(t) = \lim_{h \to 0, h < 0}(F(t+h) - F(t))/h$, $y_n(t) = nt - \ln u_n$ ($F(t)$ and a_n according to the definition in Theorem 1), $z_n = \sup_{x \geq 0} x^n/M(x)$, $n \in \mathbb{N}$, and for $t \geq a_1$ we put $G(t) = \ln P(e^t)$.

By convexity of F we get

$$F'_l(a_n) \leqslant n \leqslant F'_r(a_n), \qquad n \in \mathbb{N}.$$

This leads to the inequality

$$F(t) \geqslant F(a_n) + n(t - a_n), \qquad \forall t \in R,$$

consequently

$$na_n - F(a_n) = \sup_{t \in R} (nt - F(t))$$

and

$$e^{na_n}/M(e^{a_n}) = \sup_{x \geqslant 0} x^n/M(x). \tag{2}$$

For every $x \in R_0^+$ we have $x^n/z_n \leqslant x^n/u_n \leqslant M(x)$, and in connection with (2) it follows that $u_n = z_n$, i.e.,

$$y_n(t) = F(a_n) + n(t - a_n).$$

(The graph of y_n represents, for every $n \in \mathbb{N}$, the tangent to the graph of F at the point $(a_n, F(a_n))$ with slope n.) Now G can be written as

$$G(t) = \max_{n \in \mathbb{N}} y_n(t), \qquad t \geqslant a_1.$$

The asymptotic relation $M \sim P$ is equivalent to $\sup_{t \geqslant a_1} \{F(t) - G(t)\} < \infty$. By convexity of F we have

$$\max_{a_n \leqslant t \leqslant a_{n+1}} \{F(t) - G(t)\} = F(\bar{a}_n) - G(\bar{a}_n) = F(\bar{a}_n) - y_n(\bar{a}_n)$$

$$= F(\bar{a}_n) - F(a_n) - n(\bar{a}_n - a_n), \qquad n \in \mathbb{N},$$

where

$$\bar{a}_n = \ln(u_{n+1}/u_n) = F(a_n) - F(a_{n+1}) + (n+1)a_{n+1} - na_n$$

represents, for every $n \in \mathbb{N}$, the t-coordinate of the intersection-point of y_n and y_{n+1}; i.e., $M \sim P$ is equivalent to $\sup_{n \in \mathbb{N}} \{F(\bar{a}_n) - F(a_n) - n(\bar{a}_n - a_n)\} < \infty$.

In view of $h(x) \geqslant M(x) \geqslant P(x)$ we can replace (1) by

$$(h \sim M) \wedge (M \sim P)$$

or, as shown above, by

$$(h \sim M) \wedge \sup_{n \in \mathbb{N}} \{F(\bar{a}_n) - F(a_n) - n(\bar{a}_n - a_n)\} < \infty.$$

The assertion now follows by Theorem 1.

3. A Maximal Power Series with Non-negative Coefficients

For a given function $h\colon R_0^+ \to R^+$ which does not satisfy (1), the question arises of whether there exists a maximal power series $Q(x) \leqslant h(x)$, $x \geqslant 0$. Finally we can ask for the asymptotic growth of $Q(x)$. In connection with Theorem 2 the following result will be formulated.

Theorem 3. *Let $h\colon R_0^+ \to R^+$ be rapidly growing. There always exists a power series $Q(x) = \sum d_n x^n$, $d_n \geqslant 0$, which satisfies $Q(x) \leqslant h(x)$, $x \geqslant 0$, and is maximal in the following sense: Let $f(x) = \sum b_n x^n$, $b_n \geqslant 0$, be an arbitrary power series with $f(x) \leqslant h(x)$, $x \geqslant 0$; then it is possible to find a constant $d \in R$ such that $f(x) \leqslant dQ(x)$, $x \in R_0^+$.*

Proof. First of all note that by definition of $P(x)$ (in Theorem 2) we have $u_n = \sup_{x \geqslant 0} x^n/P(x)$; hence by Theorem 2 we obtain existence of a power series $Q(x) = \sum d_n x^n$, $d_n \geqslant 0$, with $Q \sim P$. Without loss of generality we may assume $Q(x) \leqslant h(x)$. To prove the desired property of Q let $f(x) = \sum b_n x^n$, $b_n \geqslant 0$, be an arbitrary power series with $f(x) \leqslant h(x)$, $x \in R_0^+$.

We define

$$r_n = \sup_{x \geqslant 0} x^n/f(x), \qquad n \in \mathbb{N},$$

and

$$B(x) = \max_{n \in \mathbb{N}} x^n/r_n, \qquad x \in R_0^+.$$

Again by Theorem 2 we get $f \sim B$.

Now we verify the inequality $B(x) \leqslant P(x)$, $x \geqslant 0$. Let us assume the contrary,

$$B(s) > P(s) \qquad \text{for a } s \in R_0^+. \tag{3}$$

Then there exists an integer $m \in \mathbb{N}$ with

$$s^m/r_m = B(s).$$

Hence, by (3), we obtain

$$s^m/r_m > P(s) \geqslant s^m/u_m,$$

i.e., $u_m > r_m$, which is in contradiction to $f(x) \leqslant h(x)$.

Summarizing we obtain

$$f \sim B \leqslant P \sim Q.$$

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